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ON SOME PROBLEMS CONCERNING

DIFFERENTIAL EQUATIONS OF DIFFUSION TYPE

A THESIS

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SUMMARY

The topic of this thesis originates from consideration of the classical diffusion equation

$$w_t(t,x) = A(x)w_{xx}(t,x) + B(x)w_x(t,x)$$

with elastic barrier conditions and the initial condition

$$w(0,x) = f(x), \quad a < x < b.$$

The Laplace transform with respect to time of a solution of this problem satisfies the ordinary differential equation

$$sF(x) - \left[A(x)F''(x) + B(x)F'(x)\right] = f(x).$$

The main portion of the paper is a study of a class of generalized derivative equations with certain boundary conditions. The Laplace transform equation just described is a special case of the generalized derivative equation problem, and in some cases the boundary conditions are the same.

A generalized derivative is a derivative taken with respect to a strictly increasing right continuous function. In case the function m is continuous, then the derivative of f with respect to m, $D_m f$, is defined to be

$$D_m f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{m(x + h) - m(x)},$$

provided the limit exists. The generalized derivative equation considered is

$$sF(x) - D_m D_x F(x) = f(x), \quad a < x < b, \quad s > 0,$$

with the boundary conditions

$$F(a+) = k_1 F'(a+), \quad k_1 > 0$$
$$F(b-) = - k_2 F'(b-), \quad k_2 > 0.$$

The treatment given depends on the existence of solutions of the homogeneous equation

$$sF(x) - D_m D_x F(x) = 0, \quad a < x < b.$$

Within a class of functions which are differentiable in the ordinary sense, and for which the derivative $D_m D_x$ exists and is continuous, it is shown that a unique solution satisfying given initial data exists.

Two solutions of the homogeneous equation which satisfy the boundary conditions at a and b, respectively, are used to construct a Green's function from which a unique solution of the nonhomogeneous equation is obtained.

Examples of generalized derivative equations that are related to the diffusion problem are given.

CHAPTER I

INTRODUCTION

The classical diffusion equation

$$w_t(t,x) = A(x)w_{xx}(t,x) + B(x)w_x(t,x), \tag{1}$$

for $a < x < b$, will be considered in this paper, together with certain boundary conditions which will be referred to as <u>elastic</u> <u>barrier</u> <u>conditions</u>. A one-dimensional diffusion process with elastic barrier conditions may be thought of as a process in which a particle moves in a random manner, but at an elastic barrier may be absorbed, or reflected to a point in the interior of the interval. A complete mathematical model for such phenomena has long been a problem of interest in probability theory, but apparently the development of such a model is not yet complete. However, there does appear to be some justification for the definition of the elastic barrier conditions which will be used here. The boundaries a and b will be said to be elastic (cf. W. Feller [1]), if

$$w(t,a+) = k_1 \lim_{x \to a+} \left\{ \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds \right] w_x(t,x) \right\} \tag{2}$$

and

$$w(t,b-) = -k_2 \lim_{x \to b-} \left\{ \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds \right] w_x(t,b-) \right\}, \quad (3)$$

where k_1 and k_2 are positive constants and x_0 is a fixed point in the interval $a < x < b$.

The functions A, A', and B are assumed to exist and be continuous in the interval $a < x < b$, and it is further assumed that A is positive in the same interval. Thus the integrals in equations (2) and (3) are well-defined.

A generalization of the ordinary derivative will be considered. This generalized derivative is taken with respect to a strictly increasing right continuous function. If m is continuous at x, then the derivative of f with respect to m, $D_m f$, is defined to be

$$D_m f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{m(x + h) - m(x)}.$$

The complete definition, which will include the definition of $D_m f$ at discontinuity points of m, is given in Chapter II. Equation (1) can be written in the form

$$w_t(t,x) = D_m D_u w(t,x), \quad a < x < b,$$

for suitable choice of m and u.

The rest of Chapter II is concerned with the generalized derivative equation

$$sF(x) - D_m D_x F(x) = f(x), \quad a < x < b.$$

A solution of this equation is the Laplace transform of a solution of equation (1) which satisfies the initial condition

$$w(0,x) = f(x), \quad a < x < b,$$

at least in the case where m is chosen so that $D_m D_x$ is the diffusion operator.

The existence of solutions of the homogeneous equation

$$sF(x) - D_m D_x F(x) = 0, \quad a < x < b,$$

is discussed. Two solutions of this equation which satisfy the boundary conditions at the endpoints a and b, respectively, are used to construct a Green's function, which determines the solution of

$$sF(x) - D_m D_x F(x) = f(x), \quad a < x < b,$$

with the boundary conditions

$$\begin{aligned} F(a+) &= k_1 F'(a+) \\ F(b-) &= -\, k_2 F'(b-). \end{aligned}$$

Chapter III consists of four examples, each being concerned with problems of the type discussed in Chapter II. The function m in the first example is continuous except for a jump discontinuity at the origin. The problem solved is then equivalent to one of solving

$$sF(x) - F''(x) = f(x), \; a < x < 0, \; 0 < x < b,$$

with the boundary condition

$$sF(0) - \left[F'(0+) - F'(0-)\right] = f(0).$$

This boundary condition is not of the classical type, and it is of interest to note that the resulting solution is formally analogous to solutions of the same equation treated over the separate intervals $a < x < 0$ and $0 < x < b$, respectively, when elastic barrier conditions are imposed at the origin. Examples 2 and 3 are concerned with these latter problems. Example 4 is a problem which extends Example 1 so as to include elastic barrier conditions at a and b. The result is that the solution of the problem in Example 4 may be thought of as the solution of a diffusion problem with elastic barrier conditions at both endpoints, and also at an interior point, except that a particle can pass through the interior barrier.

CHAPTER II

STUDY OF A CLASS OF GENERALIZED DERIVATIVE EQUATIONS

The Generalized Derivative.--A generalization of the ordinary derivative will now be considered. This generalized derivative will be used throughout the discussions of the diffusion problems in this paper.

Consider a strictly increasing function m defined on the finite or infinite interval $a < x < b$, and such that m is continuous from the right at each point in the interval. That is, suppose that

$$m(x) = m(x+), \quad a < x < b.$$

It will not be required that m be bounded at a or b. The right derivative of a function f with respect to m at the point x is defined to be

$$D_m^+ f(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{m(x+h) - m(x)}$$

whenever m is continuous at x, and provided the limit exists. If m is not continuous at x, then the definition is

$$D_m^+ f(x) = \frac{f(x+) - f(x-)}{m(x+) - m(x-)}.$$

The left derivative is defined to be the same as the right derivative at points where m is not continuous. At points where m is continuous

$$D_m^- f(x) = \lim_{h \to 0-} \frac{f(x + h) - f(x)}{m(x + h) - m(x)}.$$

The function f will be said to have a derivative $D_m f$ at x provided the right and left derivatives exist and are equal. A necessary condition for the existence of the derivative $D_m f$ is that f be continuous with respect to m, which is the condition that

$$\lim_{h \to 0} f(x + h) = f(x)$$

whenever

$$\lim_{h \to 0} m(x + h) = m(x).$$

Thus jumps are permitted in f at points where jumps occur in the increasing function m.

At times the use of $D_m f$ will be taken to mean $D_m^+ f$ or $D_m^- f$ if only one of these is defined, such as at the endpoints of an interval.

The Operator $D_m D_u$.--The diffusion equation (1) can be written in the form

$$w_t(t,x) = D_m D_u w(t,x), \quad a < x < b, \qquad (4)$$

by letting

$$u(x) = \int_{x_0}^{x} \exp\left[-\int_{x_0}^{y} \frac{B(s)}{A(s)}ds\right]dy$$

$$m(x) = \int_{x_0}^{x} \frac{1}{A(y)} \exp\left[\int_{x_0}^{y} \frac{B(s)}{A(s)}ds\right]dy.$$

The functions m and u thus defined are continuous because of the restrictions placed on the functions A and B. That the two equations (4) and (1) are the same will now be checked.

$$D_u w(t,x) = \lim_{h\to 0} \frac{w(t,x+h) - w(t,x)}{h\,\frac{1}{h}\int_{x}^{x+h} \exp\left[-\int_{x_0}^{y} \frac{B(s)}{A(s)}ds\right]dy}$$

$$= w_x(t,x) \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)}ds\right].$$

$D_m D_u w(t,x) =$

$$\lim_{h\to 0} \frac{w_x(t,x+h) \exp\left[\int_{x_0}^{x+h}\frac{B(s)}{A(s)}ds\right] - w_x(t,x) \exp\left[\int_{x_0}^{x}\frac{B(s)}{A(s)}ds\right]}{h\,\frac{1}{h} \int_{x}^{x+h} \frac{1}{A(y)} \exp\left[\int_{x_0}^{y} \frac{B(s)}{A(s)}ds\right]dy}$$

$$= \lim_{h\to 0} \left\{\left[\frac{1}{h} \int_{x}^{x+h} \frac{1}{A(y)} \exp\left[\int_{x_0}^{y} \frac{B(s)}{A(s)}ds\right]dy\right]^{-1} \cdot \left[\frac{w_x(t,x+h) - w_x(t,x)}{h} \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)}ds\right]\right.\right.$$

$$+ w_x(t,x) \; \frac{\exp\left[\int_{x_0}^{x+h} \frac{B(s)}{A(s)} ds\right] - \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds\right]}{h} \Bigg]\Bigg\}$$

$$= \frac{w_{xx}(t,x) \; \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds\right] + w_x(t,x) \; \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds\right]}{\frac{1}{A(x)} \exp\left[\int_{x_0}^{x} \frac{B(s)}{A(s)} ds\right]}$$

$$= A(x) \; w_{xx}(t,x) + B(x) \; w_x(t,x),$$

which is the right side of equation (1).

In connection with the operator $D_m D_u$ it may be noted that if u is continuous, then the interval may be referred to u as scale parameter. This amounts to considering the operator $D_m D_x$, as will be done in the sections to follow.

A function f defined on an interval I will be said to be in the domain of the operator $D_m D_x$ on I if

(i) the right and left derivatives f^+ and f^- exist (where

$$f^+ = D_x^+ f$$
$$f^- = D_x^- f),$$

(ii) $f^+(x) = f^+(x+) = f^-(x+)$, for x in I

$f^-(x) = f^+(x-) = f^-(x-)$, for x in I,

and

(iii) $D_m D_x^+ f$ exists and is continuous for x in I.

The definition of $D_m D_x^+ f$ implies that

$$D_m D_x^+ f = D_m D_x^- f,$$

and therefore either of these may be taken as $D_m D_x f$. The definition of the domain of the operator $D_m D_x$ just given is the one used by Feller in [2].

For a function f in the domain of the operator $D_m D_x$, there is the inversion formula

$$\int_{x_1+}^{x_2-} D_m D_x f \, dm = f^-(x_2) - f^+(x_1).$$

Such inversion formulae are discussed in articles by Daniell [3] and Jeffery [4].

<u>The Homogeneous Equation</u>.--Since it will be necessary later to find solutions of the homogeneous generalized derivative equation

$$sF(x) - D_m D_x F(x) = 0, \quad a < x < b, \tag{5}$$

the existence of such solutions will now be discussed. For the treatment of equation (5), a continuous function f will be said to be in the domain of the operator $D_m D_x$ on an interval if $D_x f$ exists and is continuous on the interval, and if $D_m D_x f$ exists and is continuous on the interval.

In order to show the existence of a solution of equation (5) satisfying the conditions

$$F(c) = k_1 \tag{6}$$

$$F'(c) = k_2, \tag{7}$$

for some c in the interval $a < x < b$, the equivalence of this problem and that of finding a solution of

$$F(x) = k_1 + k_2(x - c) + s\int_c^x dt \int_c^t F\, dm, \tag{8}$$

for $a < x < b$, is shown.

That a solution of equation (5) which satisfies the conditions (6) and (7) also satisfies equation (8) will be shown first. By using the inversion formula mentioned in the last section, it follows that if

$$D_m D_x F(x) = sF(x), \quad a < x < b$$
$$F(c) = k_1$$
$$D_x F(c) = k_2,$$

then

$$D_x F(x) = k_2 + s\int_c^x F\, dm, \quad a < x < b,$$

and finally that

$$F(x) = k_1 + k_2(x - c) + s\int_c^x dt \int_c^t F\, dm, \quad a < x < b,$$

which shows the assertion in one direction.

Now if it is assumed that F is a function which satisfies the integral equation (8), then certainly

$$F(c) = k_1$$

and since

$$D_x F(x) = k_2 + s\int_c^x F\ dm, \quad a < x < b, \tag{9}$$

it also follows that

$$D_x F(c) = k_2.$$

If now both sides of equation (9) are differentiated with respect to m, it is seen that

$$\begin{aligned} D_m D_x F(x) &= s\ D_m\int_c^x F\ dm, \quad a < x < b, \\ &= sF(x), \quad a < x < b. \end{aligned}$$

The fact that

$$D_m\int_c^x F\ dm = F(x), \quad a < x < b, \tag{10}$$

can be deduced by considering the definition of the m-derivtive and appealing to the first mean-value theorem for integrals. At points of continuity of m, the definition is

$$D_m\int_c^x F\ dm = \lim_{h\to 0} \frac{\int_x^{x+h} F\ dm}{m(x+h) - m(x)}.$$

Since F is continuous and since m increases, there is a point $x + yh$, $0 < y < 1$, such that

$$\int_x^{x+h} F\, dm = F(x + yh)\left[m(x + h) - m(x)\right].$$

Therefore

$$D_m \int_c^x F\, dm = \lim_{h \to 0} F(x + yh)$$
$$= F(x).$$

At points where m is not continuous,

$$D_m \int_c^x F\, dm = \lim_{h \to 0} \frac{\int_{x-h}^{x+h} F\, dm}{m(x+h) - m(x-h)},$$

which by the same analysis as before is

$$D_m \int_c^x F\, dm = \lim_{h \to 0} F(x + yh), \quad -1 < y < +1$$
$$= F(x).$$

Thus in either case the result (10) holds.

<u>Existence of a Solution of the Integral Equation</u>.--The existence of a solution of the integral equation

$$F(x) = k_1 + k_2(x - c) + s\int_c^x dt \int_c^t F\, dm, \qquad (11)$$

$a < x < b$, $a < c < b$, in a neighborhood $N(c,\delta)$ will be shown by the method of successive approximations. For these approximations consider

$$\phi_0(x) = k_1 + k_2(x - c)$$

$$\phi_n(x) = k_1 + k_2(x - c) + s\int_c^x dt \int_c^t \phi_{n-1}\, dm, \quad n \geq 1.$$

Now ϕ_0 exists and is continuous on any interval. If it is supposed that ϕ_{k-1} exists and is continuous for x in $N(c,\delta)$, then since m is monotonically increasing and therefore has at most a countably infinite number of jump discontinuities, it follows that ϕ_k exists and is continuous for x in $N(c,\delta)$. Therefore each ϕ_n, $n = 1,2,\cdots$, exists and is continuous in the neighborhood.

To see that the successive approximations converge to a limit function consider

$$\phi_n(x) = \phi_0(x) + \sum_{k=1}^{n}\left[\phi_k(x) - \phi_{k-1}(x)\right].$$

Note also that

$$\left|\phi_1(x) - \phi_0(x)\right| = \left| s\int_c^x dt \int_c^t \left[k_1 + k_2(y - c)\right] dm(y)\right|$$

$$\leq s\left|\int_c^x dt \int_c^t \left\{|k_1| + |k_2||(y - c)|\right\} dm(y)\right|.$$

Now since it is supposed that x is in the neighborhood $N(c,\delta)$, then

$$\left|\phi_1(x) - \phi_0(x)\right| \leq s\left[|k_1| + |k_2|\delta\right]\left|\int_c^x dt \int_c^t dm\right|$$

$$\leq s\left[|k_1| + |k_2|\delta\right]\left[\int_{c-\delta}^{c+\delta} dm\right]\delta.$$

Let

$$K = \left[|k_1| + |k_2|\delta\right]$$

and

$$q = s\delta\int_{c-\delta}^{c+\delta} dm.$$

Then

$$\left|\phi_1(x) - \phi_0(x)\right| \leq Kq,$$

for x in $N(c,\delta)$.

Now suppose that

$$\left|\phi_k(x) - \phi_{k-1}(x)\right| \leq Kq^k,$$

for some $k \geq 1$ and all x in $N(c,\delta)$. Then

$$\left|\phi_{k+1}(x) - \phi_k(x)\right| \leq \left|s\int_c^x dt \int_c^t \left|\phi_k - \phi_{k-1}\right| dm\right|$$

$$\leq \left|s\int_c^x dt \int_c^t Kq^k \, dm\right|$$

$$\leq sKq^k\left|\int_c^x dt \int_c^t dm\right| \leq Kq^k q = Kq^{k+1}.$$

The induction is complete and shows that

$$|\phi_n(x) - \phi_{n-1}(x)| \leq Kq^n,$$

for every positive integer n, for x in $N(c,\delta)$.

Therefore since the series

$$K + \sum_{n=1}^{\infty} Kq^n = K \sum_{n=0}^{\infty} q^n$$

is convergent for $q < 1$, it follows that if δ is chosen so that

$$q = s\delta \int_{c-\delta}^{c+\delta} dm < 1,$$

then the series

$$\phi_0(x) + \sum_{n=1}^{\infty} [\phi_n(x) - \phi_{n-1}(x)]$$

is uniformly convergent for x in $N(c,\delta)$ by the Weierstrass M-test. But since

$$\phi_n(x) = \phi_0(x) + \sum_{k=1}^{n} [\phi_k(x) - \phi_{k-1}(x)]$$

for all positive integers n, then the sequence $\{\phi_n\}$ converges uniformly on the interval. Moreover, the limit function is continuous since for each positive integer n, ϕ_n is continuous.

It remains to be proved that the limit function ϕ defined by

$$\phi(x) = \lim_{n \to \infty} \phi_n(x),$$

for x in $N(c,\delta)$, is a solution of the integral equation (11). For this proof it must be shown that

$$\lim_{n \to \infty} \left[k_1 + k_2(x - c) + s\int_c^x dt \int_c^t \phi_n \, dm \right]$$
$$= k_1 + k_2(x - c) + s\int_c^x dt \int_c^t \phi \, dm,$$

for x in $N(c,\delta)$. This will be so if for every $\epsilon > 0$, there exists an $N > 0$ such that

$$\left| s\int_c^x dt \int_c^t (\phi - \phi_n) \, dm \right| < \epsilon,$$

whenever $n > N$.

Certainly

$$\left| s\int_c^x dt \int_c^t (\phi - \phi_n) \, dm \right| \leq \left| s\int_c^x dt \int_c^t |\phi - \phi_n| \, dm \right|,$$

and since ϕ_n converges uniformly to ϕ for x in $N(c,\delta)$, then there is a positive integer N^* such that if $n \geq N^*$, then

$$|\phi - \phi_n| < \epsilon.$$

Thus, if $n \geq N^*$,

$$\left| s\int_c^x dt\int_c^t |\phi - \phi_n| \; dm \right| \; < \; s\left|\int_c^x dt\int_c^t dm\right| \le \epsilon q.$$

Therefore ϕ is a solution of the integral equation (11).

In order to prove the uniqueness of the solution ϕ, it will first be proved that for x in $N(c,\delta)$ the identically zero function is the only solution in case both k_1 and k_2 are zero. For if k_1 and k_2 are zero, the integral equation becomes

$$F(x) = s\int_c^x dt\int_c^t F\, dm, \quad a < x < b, \tag{12}$$

and certainly the identically zero function is a solution over any interval. Moreover, if it is supposed that a not identically zero continuous function satisfies the equation for x in $N(c,\delta)$, then for some closed interval containing c, there is a point x_M such that $|\phi(x_M)|$, which will be denoted by M, is a positive maximum for that interval. Then

$$M \le \left| s\int_c^{x_M} dt\int_c^t \phi \; dm\right| \le M\left|s\int_c^{x_M} dt\int_c^t dm\right| \le Mq < M,$$

which is a contradiction. The last of the inequalities above follows from the preceding inequality since the length of the interval, 2δ, was determined originally to make q less than one, where

$$q = s\,\delta \int_{c-\delta}^{c+\delta} dm.$$

The uniqueness of any solution of the integral equation (11) for x in $N(c,\delta)$ follows, since if ϕ_1 and ϕ_2 are two solutions, then their difference is a solution of equation (12) for x in $N(c,\delta)$, and hence is identically zero. That is

$$\phi_1(x) = \phi_2(x),$$

for x in $N(c,\delta)$.

The solution, which is now known to exist uniquely near c can be extended to any point d in the interval $a < x < b$. For, first of all, since

$$q = s\,\delta \int_{c-\delta}^{c+\delta} dm < 1,$$

or equivalently,

$$\delta < \frac{1}{s[m(c+\delta) - m(c-\delta)]},$$

then it follows that

$$\delta < \frac{1}{s[m(x_2) - m(x_1)]} = \frac{1}{sK},$$

where x_1 and x_2 are the endpoints of some closed interval containing both c and d. Thus the solution exists uniquely for x in $N(c,1/sK)$ for <u>each</u> c in the interval from x_1 to x_2.

Moreover, the solution can be extended uniquely from some other point in $N(c,\delta)$ once the functional value and slope are determined at that point. Therefore, consider a finite ordered set of points $a_1, a_2, \cdots, a_N$, such that

$$a_1 = c$$

and

$$a_N = d,$$

and such that

$$a_{i+1} - a_i < \frac{1}{sK}, \quad i = 1, 2, \cdots, N-1.$$

The initial ordinate k_1 and slope k_2 determine the ordinate and slope at a_2, and an induction argument shows that the solution can be extended uniquely to a_N, for any finite N.

<u>The Nonhomogeneous Equation</u>.--Suppose that a solution is sought for the nonhomogeneous equation

$$sF(x) - D_m D_x F(x) = f(x), \quad a < x < b, \tag{13}$$

with the boundary conditions

$$F(a+) = k_1 F'(a+), \quad k_1 > 0 \tag{14}$$

$$F(b-) = -k_2 F'(b-), \quad k_2 > 0. \tag{15}$$

Here it is supposed that the interval $a < x < b$ is finite and that m is bounded on this interval. The present use of the constants k_1 and k_2 bears no relation to the use of these symbols in the last section.

The first step in the method to be used is to find two solutions g_1 and g_2 of the homogeneous equation

$$sF(x) - D_m D_x F(x) = 0, \quad a < x < b,$$

which satisfy the boundary conditions at a and b, respectively. These solutions will then be used to construct a Green's function which will then be used to obtain the unique solution of the boundary value problem.

The solutions g_1 and g_2 certainly exist, since there exists a solution satisfying

$$F(a+) = 1$$
$$F'(a+) = 1/k_1,$$

and likewise a solution satisfying

$$F(b-) = 1$$
$$F'(b-) = -1/k_2,$$

as has already been shown. Moreover, g_1 and g_2 are determined uniquely except for a constant factor.

The Wronskian of <u>any</u> two solutions f_1 and f_2 of the homogeneous equation is given by

$$W(x) = f_2(x)f_1'(x) - f_1(x)f_2'(x), \quad a < x < b,$$

where perhaps f_1 and f_2 satisfy certain conditions at the boundaries a and b, respectively. The Wronskian can be shown to be constant by considering a certain integration by parts, as has been done in a more general case by Feller in [5]. For the present development, it suffices to consider for any x_1 and x_2 in the interval,

$$\int_{x_1}^{x_2} (f_1 D_m f_2' - f_2 D_m f_1')\, dm$$

$$= \int_{x_1}^{x_2} f_1 d(f_2') - \int_{x_1}^{x_2} f_2 d(f_1')$$

$$= f_1(x) f_2'(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f_2' d(f_1)$$

$$- f_2(x) f_1'(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f_1' d(f_2)$$

$$= W(x_1) - W(x_2).$$

The last equality above holds since f_1 and f_2 are differentiable, which implies that

$$\int_{x_1}^{x_2} f_2' d(f_1) = \int_{x_1}^{x_2} f_1' d(f_2) = \int_{x_1}^{x_2} f_1'(x) f_2'(x)\, dx.$$

Now since it is assumed that f_1 and f_2 satisfy the homogeneous equation, then

$$W(x_1) - W(x_2) = \int_{x_1}^{x_2} (f_1 D_m f_2' - f_2 D_m f_1')\, dm$$

$$= s\int_{x_1}^{x_2}(f_1f_2 - f_2f_1)dm$$

$$= 0.$$

Thus

$$W(x_1) = W(x_2)$$

for each pair of points x_1 and x_2 in the interval and hence $W(x)$ is constant over the interval.

The Green's function will be constructed from the two functions g_1 and g_2 as has already been mentioned. That the Wronskian of these two functions is not zero will now be shown. It should be recalled that g_1 satisfies the boundary condition (14) and that g_2 satisfies the boundary condition (15). Since

$$\begin{aligned} W(x) = W(a+) &= g_2(a+)g_1'(a+) - g_1(a+)g_2'(a+) \\ &= g_1'(a+)\left[g_2(a+) - k_1g_2'(a+)\right] \end{aligned}$$

and

$$\begin{aligned} W(x) = W(b-) &= g_2(b-)g_1'(b-) - g_1(b-)g_2'(b-) \\ &= - g_2'(b-)\left[g_1(b-) + k_2g_1'(b-)\right], \end{aligned}$$

it follows that the Wronskian is zero if and only if one of the two functions g_1 and g_2 satisfies <u>both</u> boundary conditions. However, suppose that there does exists a not identically zero function g which satisfies the homogeneous equation and also satisfies both boundary conditions. Then since

$$sg(x) = D_m g'(x), \quad a < x < b,$$

it follows that

$$s\int_a^b g \, dm = g'(b) - g'(a)$$

$$= -\frac{1}{k_2} g(b) - \frac{1}{k_1} g(a).$$

If g(a) or g(b) is zero, then g is necessarily the zero solution. If g is always positive, the integral on the left is positive, which is a contradiction since the right side is then negative. A similar argument applies if it is assumed that g is always negative. The only remaining ways for g to satisfy both boundary conditions are for g(a+) to be positive with g(b−) negative, or vice versa. Since g is assumed to satisfy the left boundary condition, then g'(a+) has the same sign as g(a+). Also in this case, g must change sign in the interior of the interval and actually take the value zero for some x, because of continuity. Let c be the smallest value in the interval at which g takes the value zero. Then g(x) is positive for $a < x < c$, in the event that g(a+) is positive. Now since

$$sg(x) = D_m D_x g(x), \quad a < x < b,$$

then in particular for $a < x < c$,

$$g'(x) = g'(a+) + \int_a^x g \, dm$$

which is positive for each x in the interval. Thus the fact that g and g' are positive at a, with g' remaining positive, contradicts the assumption that c is a zero of g. The same type of argument applies for g(a+) negative, in which case g'(a+) is also negative and remains negative.

Using the functions g_1 and g_2, the Green's function K is defined by

$$\begin{aligned} W(x)\, K(x,y) &= g_1(x)g_2(y), \quad \text{for } x \leq y \\ &= g_2(x)g_1(y), \quad \text{for } y \leq x. \end{aligned}$$

With the Green's function thus defined, and for f continuous and bounded on the interval $a < x < b$, the solution ϕ of the nonhomogeneous equation (13) with the boundary conditions (14) and (15) is given by

$$\phi(x) = \int_a^b K(x,y)f(y)dm(y), \quad a < x < b.$$

The condition that the function f be continuous and bounded ensures that the integral exists. That ϕ actually satisfies equation (13) is seen by a direct substitution which will now be discussed. For this discussion, the following expression for ϕ, which is a combination of the above expressions for K and ϕ, is useful:

$$\phi(x) = \frac{g_2(x)}{W}\int_a^x g_1(y)f(y)dm(y) \tag{16}$$

$$+ \frac{g_1(x)}{W}\int_x^b g_2(y)f(y)dm(y).$$

In equation (16), W is the Wronskian(which is constant).

If m has a continuous derivative with respect to x, then ϕ can be differentiated formally with respect to x to obtain

$$D_x\phi(x) =$$

$$\frac{1}{W}\, g_2(x)g_1(x)f(x)m'(x) + \frac{g_2'(x)}{W}\int_a^x g_1(y)f(y)dm(y)$$

$$-\frac{1}{W}\, g_1(x)g_2(x)m'(x)f(x) + \frac{g_1'(x)}{W}\int_x^b g_2(y)f(y)dm(y)$$

$$= \frac{g_2'(x)}{W}\int_a^x g_1(y)f(y)dm(y) \tag{17}$$

$$+ \frac{g_1'(x)}{W}\int_x^b g_2(y)f(y)dm(y) \;.$$

That the result given by this last equation is true in general when interpreted as the <u>right</u> derivative $D_x^+\phi(x)$, can be seen by performing an integration by parts on H(x), the right side of equation (17). For any points x and x_0 in the interval $a < x < b$,

$$\int_{x_0}^{x} H(x)dx = \frac{1}{W}\int_{x_0}^{x}\left[\int_{a}^{t} g_1(y)f(y)dm(y)\right]dg_2(t)$$

$$+ \frac{1}{W}\int_{x_0}^{x}\left[\int_{t}^{b} g_2(y)f(y)dm(y)\right]dg_1(t)$$

$$= \frac{g_2(t)}{W}\int_{a}^{t} g_1(y)f(y)dm(y)\Big|_{x_0}^{x}$$

$$+ \frac{g_1(t)}{W}\int_{t}^{b} g_2(y)f(y)dm(y)\Big|_{x_0}^{x}$$

$$- \frac{1}{W}\int_{x_0}^{x} g_2(t)d\left[\int_{a}^{t} g_1(y)f(y)dm(y)\right]$$

$$+ \frac{1}{W}\int_{x_0}^{x} g_1(t)d\left[\int_{b}^{t} g_2(y)f(y)dm(y)\right]$$

$$= \phi(x) - \phi(x_0).$$

Each of the integrals

$$\frac{1}{W}\int_{x_0}^{x} g_1(t)d\left[\int_{a}^{t} g_2(y)f(y)dm(y)\right]$$

and

$$\frac{1}{W}\int_{x_0}^{x} g_2(t)d\left[\int_{b}^{t} g_1(y)f(y)dm(y)\right]$$

is equal to

$$\frac{1}{W}\int_{x_0}^{x} g_1(t)g_2(t)f(t)dm(t)$$

by a change of variable theorem for the Riemann-Stieljes integral as stated by Apostol [6].

The result that H is right continuous and that

$$\int_{x_0}^{x} H(t)dt = \phi(x) - \phi(x_0)$$

for every x and x_0 in the interval will now be used to prove that

$$\phi^{+}(x) = H(x), \tag{18}$$

for all x in the interval $a < x < b$. The definition of ϕ^{+} is

$$\phi^{+}(x) = \lim_{h \to 0+} \frac{1}{h}\int_{x}^{x+h} H(t)dt,$$

provided the limit exists. At points where H is continuous, then certainly the result (18) follows. If a jump occurs in H at the point x, then in some sufficiently small interval to the right of x, H is continuous. Therefore the first mean-value theorem for integrals applies, giving

$$\begin{aligned}\phi^{+}(x) &= \lim_{h \to 0+} \frac{1}{h} H(x + yh), \quad 0 < y < 1 \\ &= H(x).\end{aligned}$$

Now since equation (18) is established, then differentiation of both sides with respect to m shows that

$$D_m\phi^+(x) = \frac{g_2'(x)g_1(x)f(x)}{W} + \frac{D_m g_2'(x)}{W}\int_a^x g_1(y)f(y)dm(y)$$

$$- \frac{g_1'(x)g_2(x)f(x)}{W} + \frac{D_m g_1'(x)}{W}\int_x^b g_2(y)f(y)dm(y)$$

$$= \frac{sg_2(x)}{W}\int_a^x g_1(y)f(y)dm(y)$$

$$+ \frac{sg_1(x)}{W}\int_x^b g_2(y)f(y)dm(y)$$

$$- \frac{g_2'(x)g_1(x) - g_1'(x)g_2(x)}{W} f(x)$$

$$= s\phi(x) - f(x),$$

which establishes the result that ϕ satisfies equation (13).

That ϕ satisfies the boundary conditions (14) and (15) follows from consideration of equations (16) and (17), and the fact that g_1 and g_2 satisfy the boundary conditions at a and b, respectively.

The uniqueness of the solution ϕ given by equation (16) follows from the fact that the difference of two solutions of two solutions of equation (13) with the boundary conditions (14) and (15) is a solution of the homogeneous equation satisfying <u>both</u> boundary conditions. It has already been shown that the only solution with this property is the identically zero solution. Therefore, ϕ is unique.

CHAPTER III

EXAMPLES OF GENERALIZED DERIVATIVE EQUATIONS

Example 1.--Consider the equation

$$sF(x) - D_m D_u F(x) = f(x), \quad s > 0,$$

on the finite interval $a < x < b$. In this example let

$$u(x) = x, \quad a < x < b \tag{19}$$

$$\begin{aligned} m(x) &= x, \quad a < x < 0 \\ &= x + 1, \quad 0 \leq x < b, \end{aligned} \tag{20}$$

so that the origin carries unit mass. Actually, the problem could be treated on the infinite interval $-\infty < x < +\infty$ as has been done by Feller in [7]. The finite interval is considered here since boundary conditions will be considered at the endpoints in a later example.

With m and u defined as above,

$$\begin{aligned} D_m D_u F(x) &= D_m F'(x) \\ &= \lim_{h \to 0} \frac{F'(x + h) - F'(x)}{(x + h) - x} \\ &= F''(x) \end{aligned}$$

when $a < x < b$ and $x \neq 0$; and also

$$\begin{aligned} D_m D_u F(0) &= D_m F'(0) \\ &= \frac{F'(0+) - F'(0-)}{m(0+) - m(0-)} \\ &= F'(0+) - F'(0-). \end{aligned}$$

Thus the problem is equivalent to one of solving

$$sF(x) - F''(x) = f(x), \quad a < x < 0,\ 0 < x < b,$$

with the boundary condition

$$sF(0) - \left[F'(0+) - F'(0-)\right] = f(0).$$

Two functions g_1 and g_2 which satisfy the homogeneous equation

$$sF(x) - F''(x) = 0, \quad a < x < 0,\ 0 < x < b,$$

and which satisfy the boundary condition

$$sF(0) - \left[F'(0+) - F'(0-)\right] = 0$$

are given by

$$\begin{aligned} g_1(x) &= e^{rx}, \quad a < x \leq 0 \\ &= e^{rx} + r \sinh(rx), \quad 0 \leq x < b, \end{aligned}$$

and

$$\begin{aligned} g_2(x) &= e^{-rx} - r \sinh(rx), \quad a < x \leq 0 \\ &= e^{-rx}, \quad 0 \leq x < b, \end{aligned}$$

where

$$r = s^{1/2}.$$

The functions g_1 and g_2 satisfy the differential equation since each is a linear combination of the solutions given by e^{rx} and e^{-rx}. Upon substitution into the boundary condition equation, it is seen for g_1 that

$$\begin{aligned} sg_1(0) &- \left[g_1'(0+) - g_1'(0-)\right] \\ &= r^2e^{r0} - \left[+re^{r0} + r^2\cosh(r0) - re^{r0}\right] \\ &= r^2 - \left[\, r + r^2 - r\right] = 0, \end{aligned}$$

and similarly for g_2 that

$$\begin{aligned} sg_2(0) &- \left[g_2'(0+) - g_2'(0-)\right] \\ &= r^2e^{-r0} - \left[-re^{-r0} + re^{-r0} + r^2\cosh(r0)\right] \\ &= r^2 - \left[-r + r^2 + r\right] = 0. \end{aligned}$$

Thus g_1 and g_2 satisfy the boundary condition.

The Wronskian, $W(x)$ is given by

$$W(x) = g_2(x)g_1'(x) - g_1(x)g_2'(x),$$

and as has already been mentioned, is constant over the interval. Hence, $W(x)$ can be calculated for $0 < x < b$. The result is

$$\begin{aligned} W(x) = e^{-rx}&\left[re^{rx} + r^2\cosh(rx)\right] \\ &-\left[e^{rx} + r^2\sinh(rx)\right] -re^{-rx} \end{aligned}$$

$$= 2r + r^2\left[e^{-rx}\cosh(rx) - e^{-rx}\sinh(rx)\right]$$
$$= 2r + r^2.$$

The Green's function K is given by

$$(2r + r^2)\ K(x,y) = g_1(x)g_2(y), \quad \text{for } x \le y$$
$$= g_2(x)g_1(y), \quad \text{for } y \le x,$$

and so for $a < x \le 0$, K is given by

$$(2r + r^2)\ K(x,y)$$
$$= e^{-rx} - r\sinh(rx)\ e^{ry}, \quad a < y \le x \le 0$$
$$= e^{rx}\ e^{-ry} + r\sinh(ry)\ , \quad a < x \le y \le 0$$
$$= e^{rx}e^{-ry}, \quad a < x \le 0 \le y < b$$

or

$$(2r + r^2)\ K(x,y)$$
$$= \tfrac{1}{2}(2 + r)e^{-r(x - y)} - \tfrac{1}{2}re^{rx}e^{ry},$$
$$\text{for } a < y \le x \le 0;$$
$$= \tfrac{1}{2}(2 + r)e^{r(x - y)} - \tfrac{1}{2}re^{rx}e^{ry},$$
$$\text{for } a < x \le y \le 0;$$
$$= \tfrac{1}{2}(2 + r)e^{r(x - y)} - \tfrac{1}{2}re^{rx}e^{-ry},$$
$$\text{for } a < x \le 0 \le y < b.$$

However, this means that when $a < x \le 0$, that for any y in the interval $a < y \not< b$,

$$(2r + r^2)\ K(x,y) = \tfrac{1}{2}(2 + r)e^{-r|x - y|} - \tfrac{1}{2}re^{rx}e^{-r|y|}$$

Similarly, for $0 \leq x < b$, the Green's function is given by

$$(2r + r^2)\ K(x,y)$$

$$\begin{aligned}
&= e^{-rx}e^{-ry}, \quad a < y \leq 0 \leq x < b \\
&= e^{-rx}\left[e^{ry} + r\ \sinh(ry)\right], \ 0 \leq y \leq x < b \\
&= \left[e^{rx} + r\ \sinh(rx)\right]e^{-ry}, \ 0 \leq x \leq y < b,
\end{aligned}$$

or for any y in the interval $a < y < b$,

$$(2r + r^2)\ K(x,y) = \tfrac{1}{2}(2 + r)e^{-r|x - y|} - \tfrac{1}{2}re^{-rx}e^{-r|y|}.$$

The solution of the problem, which is given by

$$\phi(x) = \int_a^b K(x,y)f(y)dm(y),$$

is for $a < x \leq 0$,

$$\phi(x) = \frac{1}{2r}\int_a^b e^{-r|x - y|}f(y)dy$$

$$+ \frac{e^{rx}}{2(2 + r)}\int_a^b e^{-r|y|}f(y)dy + \frac{e^{rx}f(0)}{r(2 + r)},$$

and for $0 \leq x < b$, is

$$\phi(x) = \frac{1}{2r}\int_a^b e^{-r|x - y|}f(y)dy \quad +$$

$$-\frac{e^{-rx}}{2(2+r)}\int_a^b e^{-r|y|}f(y)\,dy+\frac{e^{-rx}f(0)}{r(2+r)},$$

where in either case

$$r = s^{1/2}.$$

Example 2.--In order to shed light on the boundary condition in Example 1, which was

$$sF(0) - \left[F'(0+) - F'(0-)\right] = f(0),$$

the following problem and Example 3 will be considered.

Consider first the diffusion equation

$$w_t(t,x) = w_{xx}(t,x), \tag{21}$$

where $t \geq 0$ and $0 < x < b$. For this equation, let the left endpoint of the interval on the x-axis be an elastic barrier. That is, assign the boundary condition

$$w(t,0+) = k\, w_x(t,0+), \quad k > 0.$$

No boundary condition will be imposed at $x = b$ in this case, but let an initial condition be prescribed by

$$w(0,x) = f(x), \quad 0 < x < b.$$

If the Laplace transform of each side of the partial differential equation (21) is found, then by using the initial condition, the original problem becomes one of solving the ordinary differential equation

$$sF(x) - F''(x) = f(x), \quad 0 < x < b, \tag{22}$$

with the boundary condition

$$F(0+) = k\, F'(0+),$$

where F is the Laplace transform of w with respect to time, and s is the variable of the Laplace transform, and is positive. Note that equation (22) is of the type

$$sF(x) - D_m D_u F(x) = f(x)$$

with u(x) and m(x) identically equal to x over the interval.

The problem will be solved by the method already outlined for equations of the type (13). In doing so, a function g_1 which will satisfy the homogeneous equation and which also satisfies the boundary condition will be needed. Since $g_1(x)$ will have to be a linear combination of e^{rx} and e^{-rx}, determined up to a multiplicative constant, then g_1 will be given by

$$g_1(x) = e^{rx} + c\, e^{-rx}$$

for the proper value of the constant c. As in Example 1,

$$r = s^{1/2}.$$

Since g_1 is to satisfy the boundary condition, then

$$k(r - rc) - (1 + c) = 0$$

or

$$c = (kr - 1)/(kr + 1).$$

Thus

$$g_1(x) = e^{rx} + \frac{kr - 1}{kr + 1} e^{-rx}$$

is the desired result, since as is seen by direct substitution, it does satisfy the boundary condition.

For g_2, take

$$g_2(x) = e^{-rx}.$$

Actually, g_2 could be any solution of the homogeneous equation, but in making this choice, g_2 is a solution which is bounded at $+\infty$, if the interval were extended that far.

The Wronskian for these solutions, given by

$$W(x) = g_2(x)g_1'(x) - g_1(x)g_2'(x),$$

is

$$\begin{aligned} W(x) &= e^{-rx}(re^{-rx} + ce^{-rx}) - (e^{rx} + ce^{-rx})re^{rx} \\ &= 2r. \end{aligned}$$

The Green's function is given by

$$\begin{aligned} 2r\ K(x,y) &= e^{r(x - y)} + ce^{-r(x + y)}, \quad x \leq y \\ &= e^{-r(x - y)} + ce^{-r(x + y)}, \quad y \leq x. \end{aligned}$$

Therefore, the solution of the ordinary differential equation (22) is

$$\phi(x) = \frac{1}{2r}\int_x^b e^{r(x - y)} f(y)dy$$

$$+ \frac{1}{2r}\int_0^x e^{r(y - x)} f(y)dy$$

$$+ \frac{c}{2r}\int_0^b e^{-r(x + y)} f(y)dy,$$

which becomes

$$\phi(x) = \frac{1}{2r}\int_0^b e^{-r|x - y|} f(y)dy$$

$$+ \frac{ce^{-rx}}{2r}\int_0^b e^{-r|y|} f(y)dy,$$

where

$$r = s^{1/2}$$

and

$$c = \frac{kr - 1}{kr + 1}.$$

<u>Example 3</u>.--The following problem is like the preceding example except that the interval is $a < x < 0$ and that the elastic barrier condition is imposed at the right end, while no

condition is imposed at the left end. That is

$$w_t(t,x) = w_{xx}(t,x), \quad a < x < 0, \quad t \geq 0$$
$$w(0,x) = f(x), \quad a < x < 0$$
$$w(t,0-) = -kw_x(t,0-), \quad t \geq 0, \quad k > 0.$$

The corresponding problem after taking the Laplace transform is

$$sF(x) - F''(x) = f(x), \quad a < x < 0$$
$$F(0-) = -kF'(0-).$$

The solutions g_1 and g_2 to be used in the Green's function will be such that g_2 satisfies the right end or elastic barrier condition, and g_1 would be bounded at $-$ if the interval were extended that far. Since $g_2(x)$ will be a linear combination of e^{rx} and e^{-rx}, a representation of the form

$$g_2(x) = e^{-rx} + ce^{rx},$$

where

$$r = s^{1/2},$$

will be sought. Now if g_2 is to satisfy the boundary condition, then it must be true that

$$k(rc - r) + (1 + c) = 0,$$

from which it follows that

$$c = (kr - 1)/(kr + 1),$$

which is the same constant c as in Example 2 if k is the same. Therefore,

$$g_1(x) = e^{rx}$$
$$g_2(x) = e^{-rx} + ce^{rx},$$

for all x in the interval $a < x < 0$.

The Wronskian,

$$W(x) = g_2(x)g_1'(x) - g_1(x)g_2'(x), \quad a < x < 0,$$

is

$$\begin{aligned} W(x) &= (e^{-rx} + ce^{rx})re^{rx} - e^{rx}(-re^{-rx} + cre^{rx}) \\ &= 2r \end{aligned}$$

for all x in the interval $a < x < 0$.

The Green's function is given by

$$\begin{aligned} 2r\,K(x,y) &= e^{r(x - y)} + ce^{r(x + y)}, \quad x \leq y \\ &= e^{-r(x - y)} + ce^{r(x + y)}, \quad y \leq x. \end{aligned}$$

The solution is thus

$$\phi(x) = \frac{1}{2r}\int_a^x e^{-r(x - y)} f(y)\,dy$$

$$+ \frac{1}{2r}\int_x^0 e^{r(x - y)} f(y)\,dy \quad +$$

$$+ \frac{c}{2r}\int_a^0 e^{r(x+y)} f(y)dy,$$

or

$$\phi(x) = \frac{1}{2r}\int_a^0 e^{-r|x-y|} f(y)dy$$

$$+ \frac{ce^{rx}}{2r}\int_a^0 e^{-r|y|} f(y)dy,$$

where

$$r = s^{1/2}.$$

Example 4.--As in the previous examples consider

$$sF(x) - D_m D_u F(x) = f(x), \quad a < x < b,$$

where

$$u(x) = x, \quad a < x < b$$
$$m(x) = x, \quad a < x < 0$$
$$= x + 1, \quad 0 \leq x < b.$$

However, this time suppose that the solution is to satisfy the boundary conditions

$$F(a+) - k_1 F'(a+) = 0, \quad k_1 > 0 \qquad (23)$$
$$F(b-) + k_2 F'(b-) = 0, \quad k_2 > 0. \qquad (24)$$

In terms of ordinary derivatives, the problem is that of finding a solution of

$$sF(x) - F''(x) = f(x), \quad a < x < 0 \text{ and } 0 < x < b$$

$$sF(0) - [F'(0+) - F'(0-)] = 0$$

with the boundary conditions (23) and (24).

In the following let

$$p(x) = c_1 e^{rx} + c_2 e^{-rx}$$
$$q(x) = d_1 e^{rx} + d_2 e^{-rx},$$

where

$$r = s^{1/2}.$$

It happens that if

$$c_1 + c_2 = 1$$
$$d_1 + d_2 = 1,$$

then the functions g_1 and g_2 defined by

$$\begin{aligned} g_1(x) &= p(x), \quad a < x \leq 0 \\ &= p(x) + r \sinh(rx), \quad 0 \leq x < b \end{aligned}$$

$$\begin{aligned} g_2(x) &= q(x) - r \sinh(rx), \quad a < x \leq 0 \\ &= q(x), \quad 0 \leq x < b, \end{aligned}$$

will satisfy

$$sF(x) - F''(x) = 0, \quad a < x \leq 0 \text{ and } 0 \leq x < b$$

$$sF(0) - \left[F'(0+) - F'(0-)\right] = 0,$$

as will now be shown.

Certainly,

$$sg_i(x) - g_i''(x) = 0, \quad a < x < 0 \text{ and } 0 < x < b,$$

for i = 1,2 because g_1 and g_2 are linear combinations of the solutions given by e^{rx} and e^{-rx}. The result that g_1 and g_2 satisfy the boundary condition

$$sF(0) - \left[F'(0+) - F'(0-)\right] = 0$$

will be shown by direct substitution. For g_1, it follows that

$$r^2 p(0) - \left\{\left[p'(0+) - r^2\right] - p'(0-)\right\} = 0$$

since

$$p'(0+) = p'(0-)$$

and also

$$p(0) = c_1 + c_2 = 1.$$

Likewise, for g_2, it follows that

$$r^2 q(0) - \left\{\left[q'(0+) - r^2\right] - q'(0-)\right\} = 0$$

since

$$q'(0+) = q'(0-)$$

and

$$q(0) = d_1 + d_2 = 1.$$

Since the Wronskian of the two functions g_1 and g_2 is constant, it can be calculated for $0 < x < b$, and is given by

$$\begin{aligned} W(x) &= q(x)\Big[p'(x) + r^2\cosh(rx)\Big] \\ &\quad - \Big[p(x) + r\sinh(rx)\Big]q'(x) \\ &= \Big[q(x)\,p'(x) - p(x)q'(x)\Big] \\ &\quad + \Big[r^2 q(x)\cosh(rx) - rq'(x)\sinh(rx)\Big] \\ &= r\Big[(d_1e^{rx} + d_2e^{-rx})(c_1e^{rx} - c_2e^{-rx}) \\ &\quad - (c_1e^{rx} + c_2e^{-rx})(d_1e^{rx} - d_2e^{-rx})\Big] \\ &\quad + r^2\Big[(d_1e^{rx} + d_2e^{-rx})(\tfrac{1}{2}e^{rx} + \tfrac{1}{2}e^{-rx}) \\ &\quad - (d_1e^{rx} - d_2e^{-rx})(\tfrac{1}{2}e^{rx} - \tfrac{1}{2}e^{-rx})\Big] \\ &= r\Big[2(c_1d_2 - d_1c_2) + r\Big] \\ &= r\left(2\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + r\right). \end{aligned}$$

It now follows that if c_1, c_2, d_1, d_2 are determined such that

$$g_1(a+) = k_1 g_1'(a+)$$

and

$$g_2(b-) = -\, k_2 g(b-),$$

then the Green's function is given by

$$r\left(2\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + r\right) K(x,y)$$

$$= \left[q(x) - r\sinh(rx)\right] p(y), \quad a < y \le x \le 0$$

$$= p(x)\left[q(y) - r\sinh(ry)\right], \quad a < x \le y \le 0$$

$$= p(x)q(y), \quad a < x \le 0 \le y < b$$

for x negative, and by

$$r\left(2\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + r\right) K(x,y)$$

$$= q(x)p(y), \quad a < y \le 0 \le x < b$$

$$= q(x)\ p(y) + r\sinh(ry), \quad 0 \le y \le x < b$$

$$= \ p(x) + r\sinh(rx)\ q(y), \quad 0 \le x \le y < b,$$

when x is positive. Therefore the solution of the problem is, in terms of p and q,

$$\phi(x) = \frac{q(x) - r\sinh(rx)}{W(x)} \int_a^x p(y)f(y)\,dm(y)$$

$$+ \frac{p(x)}{W(x)}\int_x^b q(y)f(y)\,dm(y)$$

$$- \frac{rp(x)}{W(x)}\int_x^0 \sinh(ry)f(y)\,dm(y)$$

when x is negative, and is

$$\phi(x) = \frac{q(x)}{W(x)}\int_a^x p(y)f(y)dm(y)$$

$$+ \frac{p(x) + r\sinh(rx)}{W(x)}\int_x^b q(y)f(y)dm(y)$$

$$+ \frac{rq(x)}{W(x)}\int_0^x \sinh(ry)f(y)dm(y),$$

when x is positive.

All that remains is to determine c_1, c_2, d_1, d_2 so that the functions g_1 and g_2 satisfy the elastic barrier conditions at a and b, respectively. If it is assumed that g_1 satisfies

$$F(a+) = k_1F'(a+)$$

then

$$c_1e^{ra} + c_2e^{-ra} = k_1c_1re^{ra} - k_1c_2re^{-ra}$$

$$c_1(1 - k_1r)e^{ra} + c_2(1 + k_1r)e^{-ra} = 0$$

which together with the fact that

$$c_1 + c_2 = 1$$

implies that

$$c_1 = -\frac{(1 + k_1r)e^{-ra}}{(1 - k_1r)e^{ra} - (1 + k_1r)e^{-ra}} \tag{25}$$

$$c_2 = \frac{(1 - k_1 r)e^{ra}}{(1 - k_1 r)e^{ra} - (1 + k_1 r)e^{-ra}}, \tag{26}$$

provided that

$$(1 - k_1 r)e^{ra} - (1 + k_1 r)e^{-ra} \neq 0. \tag{27}$$

Likewise if it is assumed that g_2 satisfies

$$F(b-) = - k_2 F'(b-),$$

then

$$d_1 e^{rb} + d_2 e^{-rb} = - k_2 d_1 r e^{rb} + k_2 d_2 r e^{-rb}$$
$$d_1(1 + k_2 r)e^{rb} + d_2(1 - k_2 r)e^{-rb} = 0,$$

which together with the fact that

$$d_1 + d_2 = 1$$

implies that

$$d_1 = - \frac{(1 - k_2 r)e^{-rb}}{(1 + k_2 r)e^{rb} - (1 - k_2 r)e^{-rb}} \tag{28}$$

$$d_2 = \frac{(1 + k_2 r)e^{rb}}{(1 + k_2 r)e^{rb} - (1 - k_2 r)e^{-rb}}, \tag{29}$$

provided that

$$(1 + k_2 r)e^{rb} - (1 - k_2 r)e^{-rb} \neq 0. \tag{30}$$

Both of the expressions in (27) and (30) that need to be different from zero are of the form

$$\pm (1 - kr)e^{-rx} - (1 + kr)e^{rx}, \quad k > 0, \quad x > 0.$$

If one assumes that such an expression is zero for some positive value of r, then

$$e^{2rx} = \frac{1 - kr}{1 + kr},$$

which would imply that there is some positive r such that

$$e^{2rx} < 1,$$

which is impossible. Thus the constants are determined by equations (25), (26), (28), and (29) and the solution of the problem is determined.

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